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The Interior and the Exterior of the Image of the Exponential Map in Classical Lie Groups

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We give a simple description of the interior, the exterior, and the boundary of the image of the exponential map for each of the following classes of complex Lie groups: $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $O_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$ and also for each of their real forms. There is one exception: namely for special unitary groups we are not able to describe, in general, the interior and the boundary of the exponential image. In the cases $GL_n(\mathbb{R})$ and $SL_n(\mathbb{R})$ the results are due to M. Nishikawa, who has also handled the case of real orthogonal groups $O(p, q)$ when $1 \leqslant p \leqslant q \leqslant 3$.

0. INTRODUCTION

By a *classical (Lie) group* we mean one of the complex Lie groups $GL_n(\mathbb{C})$, $O_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$ or one of their real forms:

$$GL_n(\mathbb{R}), \qquad GL_n(\mathbb{H}), \qquad U(k, n-k), \qquad O(k, n-k),$$
$$O^*(n)\ n \text{ even}, \qquad Sp_{2n}(\mathbb{R}), \qquad \text{and} \qquad Sp(k, n-k).$$

If G is any Lie group, g its Lie algebra, and exp: $g \to G$ its exponential map then we shall write $E = \exp(g)$ for the image of exp in G. We say that $a \in G$ is an exponential (in G) if $a \in E$. It is clear that $aEa^{-1} = E$ for all $a \in G$, i.e., E is a union of conjugacy classes of G. The same holds for the interior, the exterior, and the boundary of E, which we denote by Int E, Ext E, and ∂E, respectively.

In an earlier paper [2] we determined which conjugacy classes of G belong to E, for each classical group G and also for the groups $SL_n(\mathbb{C})$, $SL_n(\mathbb{R})$, $SL_n(\mathbb{H})$, and $SU(k, n-k)$, which are not classical according to the definition adopted above.

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In a series of papers [8–11] M. Nishikawa studied the problem of describing Int E, Ext E, and ∂E. He solved this problem when G is $GL_n(\mathbb{R})$ or $SL_n(\mathbb{R})$, and also for groups $O(k, n-k)$ when $k, n-k \leqslant 3$.

The object of this paper is to provide an explicit and simple description of Int E, Ext E, and ∂E for all classical groups and also for the groups mentioned above, with one exception. Namely we are not able to describe, in general, Int E for the special unitary groups $SU(k, n-k)$. For the sake of completeness we also include simple proofs of Nishikawa's results for $GL_n(\mathbb{R})$ and $SL_n(\mathbb{R})$.

We assume that the reader is familiar with the classification of conjugacy classes in classical groups as presented in [1]; see also [2, 6]. In order to fix the notation we review some basic facts on conjugacy classes in Section 1.

Section 2 contains the statements of all results (Theorems 1–5) and each of the five subsequent sections contains a proof of one of these results. At the end of Section 2 we make a remark about E in the case when $G = SU(k, n-k)$.

In regard to the proofs we mention only that they rely heavily on our previous work [3–5], where we described closures of conjugacy classes in classical groups. Unfortunately the papers [2, 5] will be quoted quite often so the reader is well advised to have them at his disposal when reading the present paper.

1. Preliminaries on Conjugacy Classes

We shall view each classical group G as a triple (G, V, f), where V is a vector space over $\mathbb{F}$, $\mathbb{F}$ being the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the real quaternions $\mathbb{H}$, and f is either absent or is a non-degenerate form. If f is absent then G is the corresponding general linear group. Otherwise G is the group of all invertible linear transformations of V which preserve the form f.

For each of the 10 classes of classical groups we list in Table I the field of scalars $\mathbb{F}$, the dimension of V, the type of the form f (if any), and the signature of f (if applicable). When f is a hermitian or a real symmetric (non-degenerate) form then $\operatorname{sig}(f) = (p, q)$ means that there is an orthogonal basis of V, say $v_1, \ldots, v_p$, $w_1, \ldots, w_q$ such that $f(v_i, v_i) = 1$, $1 \leqslant i \leqslant p$, and $f(w_j, w_j) = -1$ for $1 \leqslant j \leqslant q$.

We shall denote by g the Lie algebra of a classical group $G = (G, V, f)$. Recall that g consists of all linear transformations u of V which (if f is present) satisfy $f(u(x), y) + f(x, u(y)) = 0$ for all $x, y \in V$. G acts on g by adjoint action. We say that u and u' in g are *conjugate* if $u' = a \cdot u$ for some $a \in G$, i.e., $u' = a \circ u \circ a^{-1}$.

TABLE I

No.	G	$\mathbb{F}$	dim V	f	sig(f)
1.	$GL_n(\mathbb{C})$	$\mathbb{C}$	n	—	—
2.	$GL_n(\mathbb{R})$	$\mathbb{R}$	n	—	—
3.	$GL_n(\mathbb{H})$	$\mathbb{H}$	n	—	—
4.	$U(k, n-k)$	$\mathbb{C}$	n	hermitian	$(k, n-k)$
5.	$O_n(\mathbb{C})$	$\mathbb{C}$	n	symmetric	—
6.	$O(k, n-k)$	$\mathbb{R}$	n	symmetric	$(k, n-k)$
7.	$O^*(2n)$	$\mathbb{H}$	n	skew-hermitian	—
8.	$Sp_{2n}(\mathbb{C})$	$\mathbb{C}$	$2n$	skew-symmetric	—
9.	$Sp_{2n}(\mathbb{R})$	$\mathbb{R}$	$2n$	skew-symmetric	—
10.	$Sp(k, n-k)$	$\mathbb{H}$	n	hermitian	$(k, n-k)$

If $u \in g$ then the triple (u, V, f) will be called a *Lie algebra triple*. Similarly if $a \in G$ then the triple (a, V, f) will be called a *group triple*. The isomorphisms between Lie algebra triples or group triples are defined in the usual way. The isomorphism classes of Lie algebra triples (resp. group triples) will be called *Lie algebra types* (resp. *group types*). These types were introduced by Bourgoyne and Cushman [1] in order to give a concise description of the conjugacy classes in classical groups and their Lie algebras.

Let Δ and Δ' be Lie algebra types of the same kind, i.e., corresponding to the same row of our Table I. If $(u, V, f) \in \Delta$ and $(u', V', f') \in \Delta'$ then we define the sum $\Delta + \Delta'$ to be the Lie algebra type containing the Lie triple $(u \oplus u', V \oplus V', f \oplus f')$.

One defines similarly the sum $\Gamma + \Gamma'$ of two group types Γ and Γ' of the same kind.

If $(u, V, f) \in \Delta$, where Δ is a Lie type, then we define $\dim \Delta = \dim V$, $\mathrm{eig}(\Delta) = \mathrm{eig}(u)$, where $\mathrm{eig}(u)$ is the set of eigenvalues of u, and $\mathrm{sig}(\Delta) = \mathrm{sig}(f)$ if f is present and has a signature. Similarly if $(a, V, f) \in \Gamma$, where Γ is a group type, we may define Γ^k $(k \in \mathbb{Z})$ to be the group type containing (a^k, V, f), etc.

If t lies in the center of $\mathbb{F}$ and $(u, V, f) \in \Delta$ then we define $t\Delta$ to be the Lie type containing (tu, V, f). The exponential map exp: $g \to G$ gives rise to a map exp from Lie algebra types to group types: if $(u, V, f) \in \Delta$ then $\Gamma := \exp \Delta$ means that $(\exp(u), V, f) \in \Gamma$. We say that a group type Γ is *an exponential* if $\Gamma = \exp \Delta$ for some Lie algebra type Δ (of the same kind).

We say that a Lie type Δ (resp. group type Γ) is the *zero type* if dim $\Delta = 0$ (resp. $\dim \Gamma = 0$). A Lie type Δ is *indecomposable* if $\Delta \neq 0$ and $\Delta = \Delta' + \Delta''$ implies that $\Delta' = 0$ or $\Delta'' = 0$. One defines similarly indecomposability for group types.

Every Lie type (or group type) can be uniquely written as a sum of indecomposable types; see [1]. Hence in order to describe the conjugacy classes in classical groups or their Lie algebras it suffices to list all indecomposable group types and indecomposable Lie types. The list of indecomposable Lie algebra types is given in [1], and was reproduced in our paper [2], where we also added the list of indecomposable group types.

We say that a group type Γ *belongs* to a classical group (G, V, f) if there exists $a \in G$ such that $(a, V, f) \in \Gamma$. There is a bijective correspondence between the group types that belong to G and the conjugacy classes of G. It will be convenient to denote by Γ also the conjugacy class of G which corresponds to the group type Γ.

Let Γ and Γ' be group types belonging to a classical group G. We shall write $\Gamma \geqslant \Gamma'$ if the conjugacy class Γ' of G lies in the closure of the conjugacy class Γ. The set of all conjugacy classes of G is partially ordered by the relation $\geqslant$. We shall often make use of this relation in the following way. Assume that we want to prove that some conjugacy class Γ' of G lies in the boundary ∂E of the image E of the exponential map exp: $g \to G$. If $\Gamma \geqslant \Gamma'$ and $\Gamma \subset \partial E$ then it follows that $\Gamma' \subset \partial E$. Indeed this follows from the fact that ∂E is closed.

We now introduce some matrix notation which will be used quite often in the sequel.

By I_n we denote the identity matrix of size n and by $J_n(\lambda)$ the Jordan block

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}$$

of size n. In particular we set $N_n = J_n(0)$.

By S_n we denote the matrix

$$S_n = \begin{bmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ \iddots & & & & \end{bmatrix}$$

of size n which has alternating $+1$ and -1 entries on the side diagonal and zeros elsewhere.

If A and $B=(b_{ij})$ are matrices then their *tensor product* $A \otimes B$ is the block matrix (Ab_{ij}), and their *direct sum* $A \oplus B$ is the block matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

For any real or complex matrix A we denote by A' (resp. A^*) the transpose (resp. conjugate transpose) of A.

Furthermore if A is square of order n and $\lambda \in \mathbb{C}$ is one of its eigenvalues then we define the *multiplicity* of λ to be its multiplicity as a root of the characteristic polynomial of A. In that case we define the *generalized eigenspace* of A for eigenvalue λ to be the nullspace of $(A-\lambda I_n)^m$, where m is the multiplicity of λ.

We say that A is *non-derogatory* if its minimal and characteristic polynomials coincide.

The diagonal matrix of size n whose successive diagonal entries are $\lambda_1, ..., \lambda_n$ will be written as

$$\operatorname{diag}(\lambda_1, ..., \lambda_n).$$

2. Statement of the Results

Let (G, V, f) be a classical group, g its Lie algebra, $\exp: g \to G$ its exponential map and $E = \exp(g) \subset G$. We denote by $\operatorname{Int} E$, $\operatorname{Ext} E$ and ∂E the interior, the exterior and the boundary of E for the ordinary topology of G (G is viewed as a real Lie group).

Our main problem is to describe these three subsets of G. (Since $\operatorname{Int} E$, $\operatorname{Ext} E$ and ∂E form a partition of G, it suffices to determine only two of them.)

The groups $SL_n(\mathbb{C})$, $SL_n(\mathbb{R})$, $SL_n(\mathbb{H})$ and $SU(k, n-k)$ will be included in our discussion (although they are not classical by the definition adopted in this paper.)

It is known, see [2], that $E = G$ if G is one of the groups: $GL_n(\mathbb{C})$, $GL_n(\mathbb{H})$, $SL_n(\mathbb{H})$, $U(k, n-k)$, $O^*(2n)$, or $Sp(k, n-k)$. Hence these cases will be dismissed from further discussion.

The remaining cases are: $SL_n(\mathbb{C})$, $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $SU(k, n-k)$, $O_n(\mathbb{C})$, $O(k, n-k)$, $Sp_{2n}(\mathbb{C})$, and $Sp_{2n}(\mathbb{R})$.

Theorem 1. *Let G be one of the groups $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, or $Sp_{2n}(\mathbb{R})$. Then*

(i) Int E *is the set of all matrices in* G *which have no negative eigenvalues;*

(ii) Ext E *is the set of all matrices in* G *which have at least one negative eigenvalue of odd multiplicity.*

In the case when $G = GL_n(\mathbb{R})$ or $SL_n(\mathbb{R})$ this result is due to Nishikawa [8, 9].

THEOREM 2. *Let* $G = O(p, q)$ *and* $a \in G$. *Then*

(i) $a \in \operatorname{Int} E$ *if and only if* a *has no negative eigenvalue* $\lambda < -1$ *and the generalized eigenspace of* a *for eigenvalue* -1 *is (positive or negative) definite;*

(ii) $a \in \operatorname{Ext} E$ *if and only if either* a *has at least one negative eigenvalue of odd multiplicity or else* -1 *is an eigenvalue of* a *of even multiplicity and both components of the signature of the generalized eigenspace of* a *for eigenvalue* -1 *are odd integers.*

THEOREM 3. *Let* G *be one of the groups* $O_n(\mathbb{C})$ *or* $Sp_{2n}(\mathbb{C})$. *Then the closure of* E *is the identity component of* G. *The interior of* E *consists of all matrices in* G *for which* -1 *is not an eigenvalue and, in the case when* $G = O_n(\mathbb{C})$, *the matrices for which* -1 *is an eigenvalue of multiplicity* 2.

In order to state our result for $SL_n(\mathbb{C})$ we need some more notation. Let $a \in SL_n(\mathbb{C})$ and let

$$J_{m_1}(\lambda_1) \oplus \cdots \oplus J_{m_u}(\lambda_u)$$

be the Jordan canonical form of a. Then we define $p(a) \in \mathbb{Z}$ by

$$p(a) = GCD(m_1, ..., m_u).$$

For each k, $1 \leqslant k \leqslant u$, choose $\theta_k \in \mathbb{R}$ such that

$$\lambda_k = |\lambda_k| \exp(i\theta_k).$$

Note that we may have $\lambda_k = \lambda_l$ and $\theta_k \neq \theta_l$ for some $k \neq l$. Since $\det(a) = 1$, it follows that

$$m_1\theta_1 + \cdots + m_u\theta_u = 2\pi s$$

for some $s \in \mathbb{Z}$. Since each θ_k is unique modulo $2\pi\mathbb{Z}$, it follows that s is unique modulo $p(a)$. Hence we may define $q(a) \in \mathbb{Z}/p(a)\,\mathbb{Z}$ by

$$q(a) \equiv s \qquad \operatorname{mod} p(a).$$

Theorem (1.7) of [2] implies that a is an exponential in $SL_n(\mathbb{C})$ if and only if $q(a)=0$.

Recall that a matrix is called *non-derogatory* (or cyclic) if its characteristic and minimal polynomials coincide. Clearly there exists a non-derogatory matrix $\tilde{a}$ having the same characteristic polynomial as a. Moreover $\tilde{a}$ is unique up to conjugacy in $SL_n(\mathbb{C})$ and so we can define $\tilde{p}(a) \in \mathbb{Z}$ and $\tilde{q}(a) \in \mathbb{Z}/\tilde{p}(a)\,\mathbb{Z}$ by

$$\tilde{p}(a)=p(\tilde{a}), \qquad \tilde{q}(a)=q(\tilde{a}).$$

Now we can state our theorem.

THEOREM 4. *If $G=SL_n(\mathbb{C})$ then E is dense in G and*

$$\operatorname{Int} E=\{a \in G: \tilde{q}(a)=0\}.$$

Finally let $G=SU(k, n-k)$. In this case our result is incomplete: we obtain a simple description of Ext E but, in general, we do not know how to describe Int E.

THEOREM 5. *If $G=SU(k, n-k)$, $0 \leqslant k \leqslant n-k$, then* Ext E *consists of all matrices $a \in G$ which satisfy the following two conditions:*

(a) *a has no eigenvalues on the unit circle.*

(b) *If $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of a inside the unit circle and if $m_1, \ldots, m_k$ are their respective multiplicities then the integer*

$$s:=\frac{1}{\pi}\left(m_1 \arg \lambda_1+\cdots+m_k \arg \lambda_k\right)$$

is odd.

In particular, if $k \neq n-k$ then Ext $E=\varnothing$.

In conclusion we make a remark about Int E when $G=SU(k, n-k)$.

If $a \in G$ has no eigenvalues on the unit circle (this implies that $2k=n$) then $a \in \operatorname{Int} E$ if and only if the integer s defined in condition (b) of Theorem 5 is divisible by the $GCD(m_1, \ldots, m_k)$.

3. Proof of Theorem 1

We start with a sequence of four lemmas.

LEMMA(3.1). *Let $G = GL_{2m}(\mathbb{R})$ and let $\lambda < 0$. Then the conjugacy class $\Gamma_{2m-1}(\lambda)$ of G lies in the closure of the union of the conjugacy classes*

$$\Gamma_{m-1}(\lambda e^{i\theta}, \lambda e^{-i\theta}), \qquad 0 < \theta < \pi.$$

In particular, $\Gamma_{2m-1}(\lambda) \subset \bar{E}$.

Proof. For $0 < \theta < \pi$ let

$$C_\theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \otimes I_m + \begin{pmatrix} 0 & 0 \\ \theta & 0 \end{pmatrix} \otimes N_m.$$

The eigenvalues of C_θ are $i\theta$ and $-i\theta$ and each of them has multiplicity m. Since

$$\operatorname{rank}(C_\theta^2 + \theta^2 I_{2m}) = 2m - 2,$$

it follows that the elementary divisors of C_θ are $(X - i\theta)^m$ and $(X + i\theta)^m$.

Let $D_\theta = \operatorname{diag}(1, \theta, \theta^2, ..., \theta^{2m-1})$ and $B_\theta = D_\theta C_\theta D_\theta^{-1}$. It is easy to verify that

$$\lim_{\theta \to 0+} B_\theta = N_{2m}.$$

Now define φ: $(0, \pi) \to G$ by $\varphi(\theta) = \lambda \exp B_\theta$. Clearly φ is continuous,

$$\varphi(\theta) \in \Gamma(\lambda e^{i\theta}, \lambda e^{-i\theta}), \qquad 0 < \theta < \pi,$$

and

$$\lim_{\theta \to 0+} \varphi(\theta) = \lambda \exp N_{2m} \in \Gamma_{2m-1}(\lambda).$$

This proves the first assertion of the lemma. The second is a consequence of the first and the fact that $\Gamma(\lambda e^{i\theta}, \lambda e^{-i\theta}) \subset E$ for $0 < \theta < \pi$; see [2].

LEMMA(3.2). *If $G = Sp_{4m}(\mathbb{R})$ and $\lambda < -1$ then $\Gamma_{2m-1}(\lambda, \lambda^{-1}) \subset \bar{E}$.*

Proof. We may assume that the matrix of the skew-symmetric form defining G is

$$J = \begin{pmatrix} 0 & I_{2m} \\ -I_{2m} & 0 \end{pmatrix}.$$

We have an embedding $GL_{2m}(\mathbb{R}) \to G$ given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A'^{-1} \end{pmatrix}.$$

Under this embedding, the conjugacy class $\Gamma_{m-1}(\lambda e^{i\theta}, \lambda e^{-i\theta})$, $0<\theta<\pi$, of $GL_{2m}(\mathbb{R})$ is mapped into the conjugacy class $\Gamma_{m-1}(\lambda e^{i\theta}, \lambda e^{-i\theta}, \lambda^{-1}e^{i\theta}, \lambda^{-1}e^{-i\theta})$ of G, and the conjugacy class $\Gamma_{2m-1}(\lambda)$ is mapped into $\Gamma_{2m-1}(\lambda, \lambda^{-1})$.

Now the assertion $\Gamma_{2m-1}(\lambda, \lambda^{-1}) \subset \bar{E}$ follows from Lemma (3.1).

LEMMA (3.3). *If $G = Sp_{4m}(\mathbb{R})$ and $\varepsilon = \pm$ then $\Gamma^{\varepsilon}_{4m-1}(-1) \subset \bar{E}$.*

Proof. We view G as a triple (G, V, f). Choose a basis of V such that the matrix of f with respect to this basis is

$$J = \varepsilon \begin{pmatrix} 0 & I_{2m} \\ -I_{2m} & 0 \end{pmatrix}.$$

For $t \in \mathbb{R}$ let

$$B_t = \begin{pmatrix} J_{2m}(t) & I_{2m} \\ 0 & -J_{2m}(t)' \end{pmatrix},$$

and note that $B_tJ + JB'_t = 0$. It is easy to check that $B_0^{4m-1} \neq 0$ and consequently B_0 has only one elementary divisor, namely X^{4m}. On the other hand if $t \neq 0$ then B_t has two elementary divisors, namely $(X-t)^{2m}$ and $(X+t)^{2m}$.

Let $A_t = -\exp B_t$. This defines a continuous map $\mathbb{R} \to G$ sending $t \mapsto A_t$. If $t \neq 0$ then $A_t \in \Gamma_{2m-1}(-e^t, -e^{-t})$ while $A_0 \in \Gamma^{\varepsilon}_{4m-1}(-1)$.

The assertion of the lemma now follows from Lemma (3.2).

LEMMA (3.4). *If $G = Sp_{4m+2}(\mathbb{R})$ and $\varepsilon = \pm$ then $\Gamma^{\varepsilon}_{4m+1}(-1) \subset \bar{E}$.*

Proof. We shall view G as a triple (G, V, f). Choose a basis of V such that the matrix of f with respect to this basis is

$$J = \varepsilon \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes S_{2m+1}.$$

For $0<\theta<\pi$ let

$$C_\theta = \begin{pmatrix} C_1 & C_2 & 0 \\ 0 & C_3 & C_4 \\ 0 & 0 & C_5 \end{pmatrix},$$

where

$$C_1 = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \otimes I_m + \begin{pmatrix} 1 & 0 \\ \theta^{-1} & 1 \end{pmatrix} \otimes N_m,$$

$$C_3 = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix},$$

$$C_5 = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \otimes I_m + \begin{pmatrix} 1 & 0 \\ -\theta^{-1} & 1 \end{pmatrix} \otimes N_m,$$

and all entries of C_2 (resp. C_4) are zero except the lower left-hand block of size 2, which is

$$\begin{pmatrix} 1 & 0 \\ \theta^{-1} & 1 \end{pmatrix} \quad \left(\text{resp.} \begin{pmatrix} 1 & 0 \\ -\theta^{-1} & 1 \end{pmatrix}\right).$$

It is easy to check that $C_\theta J + JC'_\theta = 0$. An easy computation shows that $\operatorname{rank}(C_\theta^2 + \theta^2 I_{4m+2}) = 4m$. Since the eigenvalues of C_θ are $i\theta$ and $-i\theta$, each with multiplicity $2m+1$, it follows that the elementary divisors of C_θ are $(X - i\theta)^{2m+1}$ and $(X + i\theta)^{2m+1}$.

The diagonal matrix

$$D_\theta = \theta^{-1/2} \cdot \operatorname{diag}(1, \theta, 1, \theta, \ldots, 1, \theta)$$

of size $4m+2$ satisfies $D_\theta J D'_\theta = J$ and so $D_\theta \in G$.

One can easily verify that the matrix $B_\theta = D_\theta C_\theta D_\theta^{-1}$ has a limit, say B_0, as $\theta \to 0+$ and B_0 is nilpotent and has only one elementary divisor namely X^{4m}.

Now define $A_\theta = -\exp B_\theta$, $0 < \theta < \pi$. Then $A_\theta \in G$ and belongs to the conjugacy class $\Gamma^{\varepsilon}_{2m}(-e^{i\theta}, -e^{-i\theta})$; in particular $A_\theta \in E$. Since

$$\lim_{\theta \to 0+} A_\theta = -\exp B_0 \in \Gamma^{\varepsilon}_{4m+1}(-1),$$

our assertion is proved.

We now begin the proof of Theorem 1.

Let Ω (resp. Φ) be the set of all matrices in G which have no negative eigenvalues (resp. which have at least one negative eigenvalue of odd multiplicity). It is clear that both Ω and Φ are open in G. From [2] we conclude that $\Omega \subset E$ and $\Phi \cap E = \varnothing$. Consequently, we have $\Omega \subset \operatorname{Int} E$ and $\Phi \subset \operatorname{Ext} E$. Let $\Delta = G \backslash (\Omega \cup \Phi)$. It remains to show that $\Delta \subset \partial E$.

Let $a \in \Delta$. We have to show that $a \in \partial E$. From the definition of Δ we know that a has at least one negative eigenvalue and that each negative eigenvalue of a has even multiplicity.

We shall now distinguish two cases.

Case 1. $G = GL_n(\mathbb{R})$ or $SL_n(\mathbb{R})$. Let Γ be the conjugacy class of G containing a. There is a unique conjugacy class Γ' of G such that if $b \in \Gamma'$ then a and b have the same eigenvalues with the same multiplicities and moreover for each eigenvalue λ the matrix b has only one elementary divisor which is a power of $X - \lambda$. Since $\Gamma' \geqslant \Gamma$, in order to prove that $\Gamma \subset \partial E$ it suffices to show that $\Gamma' \subset \partial E$, i.e., we may assume that $\Gamma = \Gamma'$.

Since $a \in \Gamma = \Gamma'$ has a negative eigenvalue, it follows from [2] that $a \notin E$. Hence we need only show that $a \in \bar{E}$. Since $\Gamma = \Gamma'$ and each negative eigenvalue of a has even multiplicity, the assertion $a \in \bar{E}$ is an immediate consequence of Lemma (3.1).

Case 2. $G = Sp_{2n}(\mathbb{R})$. We view G as a triple (G, V, f). Let Γ be the group type containing the triple (a, V, f). In order to show that $a \in \partial E$ it suffices to prove this in two special cases only:

$$\operatorname{eig}(\Gamma) = \{\lambda, \lambda^{-1}\}, \lambda < -1; \tag{3.5}$$

and

$$\operatorname{eig}(\Gamma) = \{-1\}. \tag{3.6}$$

In case (3.5) the multiplicity n of λ is even (since $a \in \Delta$), say $n = 2m$. In that case we have $\Gamma_{n-1}(\lambda, \lambda^{-1}) \geqslant \Gamma$, while in case (3.6) we have $\Gamma^{\varepsilon}_{2n-1}(-1) \geqslant \Gamma$ for some $\varepsilon = \pm$. Consequently it suffices to consider only the following possibilities for Γ:

$$\Gamma_{2m-1}(\lambda, \lambda^{-1}), \qquad \lambda < -1; \Gamma^{\varepsilon}_{2n-1}(-1), \varepsilon = \pm.$$

In these cases $\Gamma \cap E = \varnothing$ (where Γ denotes the conjugacy class of G which corresponds to the group type Γ). Hence it remains to show that $\Gamma \subset \bar{E}$.

If $\Gamma = \Gamma_{2m-1}(\lambda, \lambda^{-1})$, $\lambda < -1$, this was shown in Lemma (3.2). If $\Gamma = \Gamma^{\varepsilon}_{2n-1}(-1)$ then this claim was proved in Lemma (3.3) for n even and in Lemma (3.4) for n odd.

4. Proof of Theorem 2

We prove first two lemmas.

LEMMA (4.1). *Let (G, V, f) be the group $O(p, q)$, $\operatorname{sig}(f) = (p, q)$, $p = q + 1$, q even, say $q = 2m$. Then the conjugacy class $\Gamma^{+}_{2q}(-1)$ of G lies in the closure of the union of the conjugacy classes*

$$\Gamma^{\varepsilon}_{q-1}(-e^{i\theta}, -e^{-i\theta}) + \Gamma^{+}_{0}(-1), \qquad 0 < \theta < \pi, \varepsilon = \pm.$$

Proof. Choose a basis of V so that the matrix of f with respect to this basis is $S = S_{4m+1}$. For $0 < \theta < \pi$ define

$$C = \begin{pmatrix} C_1 & v' & C_2 \\ 0 & 0 & w \\ 0 & 0 & C_1 \end{pmatrix}$$

where

$$C_1 = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \otimes I_m + \begin{pmatrix} 1 & 0 \\ \theta^{-1} & 1 \end{pmatrix} \otimes N_m,$$

$$v = (0, ..., 0, \theta^{-1}),$$

$$w = (\theta^{-1}, 0, ..., 0),$$

and C_2 has all entries zero except for the block of size 2 in the lower left hand corner which is

$$\begin{pmatrix} 1 & 0 \\ \theta^{-1} & 1 \end{pmatrix}.$$

We have $C_\theta S + SC'_\theta = 0$ and the eigenvalues of C_θ are 0, $i\theta$, and $-i\theta$ with respective multiplicities 1, q, and q. An easy computation shows that

$$\operatorname{rank}(C_\theta^2 + \theta^2 I_{2q+1}) = 2q - 1,$$

and consequently the elementary divisors of C_θ are X, $(X - i\theta)^q$, and $(X + i\theta)^q$.

If

$$D_\theta = \operatorname{diag}(\underbrace{1, \theta, ..., 1, \theta}_{q}, 1, \underbrace{\theta^{-1}, 1, ..., \theta^{-1}, 1}_{q})$$

then $D_\theta S D'_\theta = S$, i.e., $D_\theta \in G$.

Let $B_\theta = D_\theta C_\theta D_\theta^{-1}$. One checks easily that $\lim_{\theta \to 0+} B_\theta$ exists and that this limit, say B_0, has only one elementary divisor, namely X^{2q+1}.

Now define $A_\theta = -\exp B_\theta$, $0 < \theta < \pi$. It follows that $A_\theta \in G$ and

$$A_\theta \in \Gamma^{\varepsilon}_{q-1}(-e^{i\theta}, -e^{-i\theta}) + \Gamma^{+}_{0}(-1)$$

for some $\varepsilon = \pm$.

Since $\lim_{\theta \to 0+} A_\theta = -\exp B_0$ belongs to $\Gamma^{+}_{2q}(-1)$, the lemma is established.

LEMMA (4.2). *Let (G, V, f) be the group $O(p, q)$, $\operatorname{sig}(f) = (p, q)$, $p = q + 1$, q odd, say $q = 2m + 1$. Then the conjugacy class $\Gamma_{2q}^{+}(-1)$ of G lies in the closure of the union of the conjugacy classes*

$$\Gamma_{q-1}^{+}(-e^{i\theta}, -e^{-i\theta}) + \Gamma_{0}^{-}(-1), \qquad 0 < \theta < \pi.$$

Proof. Choose a basis of V so that the matrix of f with respect to that basis is $S = S_{4m+3}$. For $0 < \theta < \pi$ define

$$C = \begin{pmatrix} C_1 & C_2 & 0 \\ 0 & C_3 & C_4 \\ 0 & 0 & C_1 \end{pmatrix},$$

where

$$C_1 = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \otimes I_{m-1} + \begin{pmatrix} 0 & 0 \\ \theta^{-1} & 0 \end{pmatrix} \otimes N_{m-1},$$

$$C_3 = \begin{pmatrix} \theta & \theta & 0 \\ -\theta & 0 & \theta \\ 0 & -\theta & -\theta \end{pmatrix},$$

and all the entries of C_2 and C_4 are zeros except the entries in the lower left-hand corner which are equal to θ^{-1}.

Then $C_\theta S + SC'_\theta = 0$ and the eigenvalues of C are 0, $i\theta$, and $-i\theta$ with respective multiplicities 1,q, and q. An easy computation shows that

$$\operatorname{rank}(C_\theta^2 + \theta^2 I_{2q+1}) = 2q - 1,$$

and consequently the elementary divisors of C_θ are X, $(X - i\theta)^q$, and $(X + i\theta)^q$.

Let D_θ be the diagonal matrix

$$D_\theta = \operatorname{diag}(\underbrace{\theta^{-1}, 1, \ldots, \theta^{-1}}_{q}, 1, \underbrace{\theta, \ldots, 1, \theta}_{q}),$$

and $B_\theta = D_\theta C_\theta D_\theta^{-1}$. Since $D_\theta S D'_\theta = S$, we have $D_\theta \in G$. Furthermore we have

$$\lim_{\theta \to 0+} B_\theta = N_{2q+1}.$$

The matrix $A_\theta = -\exp B_\theta$ belongs to G for $0<\theta<\pi$ and has elementary divisors $X+1$, $(X+e^{i\theta})^q$, and $(X+e^{-i\theta})^q$. Consequently

$$A_\theta \in \Gamma_{q-1}^{+}(-e^{i\theta}, -e^{-i\theta}) + \Gamma_0^{-}(-1), \qquad 0<\theta<\pi.$$

Since $\lim_{\theta\to 0+} A_\theta = -\exp B_0$ belongs to $\Gamma_{2q}^{+}(-1)$, the assertion of the lemma is proved.

We now begin the proof of Theorem 2.

Denote by Ω the set of all matrices $a \in G$ which have no negative eigenvalues except perhaps the eigenvalue -1 in which case the generalized eigenspace of a for eigenvalue -1 is definite.

Denote by Φ the set of all $a \in G$ satisfying at least one of the following two conditions:

(a) a has a negative eigenvalue of odd multiplicity;

(b) -1 is an eigenvalue of a of even multiplicity and both components of the signature of the generalized eigenspace for that eigenvalue are odd.

It is easy to see that both Ω and Φ are open in G. From [2] we know that $\Omega \subset E$ and $\Phi \cap E = \emptyset$. Consequently we have $\Omega \subset \operatorname{Int} E$ and $\Phi \subset \operatorname{Ext} E$. Let $\Delta := G\backslash(\Omega \cup \Phi)$. It remains to show that $\Delta \subset \partial E$.

Thus let $a \in \Delta$. By definition of Δ we know that a has at least one negative eigenvalue, that all negative eigenvalues of a have even multiplicities and if -1 is an eigenvalue of a then both components of the signature of the generalized eigenspace of a for eigenvalue -1 are even and, in the case that -1 is the only negative eigenvalue of a, this eigenspace is indefinite.

Recall that we view G as a triple (G, V, f). Let Γ be the group type containing the triple (a, V, f) and let Γ also denote the conjugacy class of G containing a. In order to prove that $\Gamma \subset \partial E$, it suffices to consider only the following two special cases:

$$\operatorname{eig}(\Gamma) = \{\lambda, \lambda^{-1}\}, \qquad \lambda < -1, \tag{4.3}$$

and

$$\operatorname{eig}(\Gamma) = \{-1\}. \tag{4.4}$$

We deal first with the case (4.3). Since $\Gamma \subset \Delta$, we know that the multiplicity of the eigenvalue λ of Γ is even, say $2m$. Since $\Gamma_{2m-1}(\lambda, \lambda^{-1}) \geqslant \Gamma$, we may assume that in fact $\Gamma = \Gamma_{2m-1}(\lambda, \lambda^{-1})$.

Fix a basis of V such that the matrix of f with respect to that basis is

$$\begin{pmatrix} 0 & I_{2m} \\ I_{2m} & 0 \end{pmatrix}.$$

Then we have an embedding $GL_{2m}(\mathbb{R}) \to G$ given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A'^{-1} \end{pmatrix}.$$

By using Lemma (3.1) we conclude that Γ is contained in the closure of the union of conjugacy classes

$$\Gamma_{m-1}(\lambda e^{i\theta}, \lambda e^{-i\theta}, \lambda^{-1} e^{i\theta}, \lambda^{-1} e^{-i\theta}), \qquad 0<\theta<\pi.$$

Since this union is contained in E, we have $a \in \bar{E}$. Since $a \in \Gamma_{2m-1}(\lambda, \lambda^{-1})$, we have $a \notin E$ and consequently $a \in \partial E$.

Next we consider the case (4.4). We have $\operatorname{sig}(f) = (p, q)$, where both p and q are even and non-zero. We may assume that $p \geqslant q$. By using [5] we conclude that

$$\Gamma_{2q}^{+}(-1) + (p-q-1)\,\Gamma_0^{+}(-1) \geqslant \Gamma \qquad \text{if} \quad p>q$$

and

$$\Gamma_{p+q-2}^{+}(-1) + \Gamma_0^{-}(-1) \geqslant \Gamma \qquad \text{if} \quad p=q.$$

Consequently, we may assume that

$$\Gamma = \begin{cases} \Gamma_{2q}^{+}(-1) + (p-q-1)\,\Gamma_0^{+}(-1) & \text{if} \quad p>q, \\ \Gamma_{2p-2}^{+}(-1) + \Gamma_0^{-}(-1) & \text{if} \quad p=q. \end{cases}$$

It follows from Lemma (4.1) that if $p>q$ then Γ lies in the closure of the union of conjugacy classes

$$\Gamma_{q-1}^{\varepsilon}(-e^{i\theta}, -e^{-i\theta}) + (p-q)\,\Gamma_0^{+}(-1), \qquad 0<\theta<\pi, \varepsilon = \pm.$$

Since p and q are even, this union is contained in E and consequently $\Gamma \subset \bar{E}$. On the other hand Γ is not an exponential and consequently $a \in \Gamma \subset \partial E$.

If $p=q$ then a similar argument using Lemma (4.2) instead of Lemma (4.1) shows that $a \in \partial E$.

This completes the proof of Theorem 2.

5. Proof of Theorem 3

The first assertion is a consequence of a theorem of Borel which asserts that E is dense in G for every connected complex Lie group G; see [7].

Let Ω be the set of all matrices in G for which -1 is not an eigenvalue and, in the case $G = O_n(\mathbb{C})$, the matrices for which -1 is an eigenvalue of multiplicity 2. It is clear that Ω is open in G. It follows from [2] that $\Omega \subset E$, and consequently $\Omega \subset \operatorname{Int} E$. Let $\Delta = G_0 \backslash \Omega$, where G_0 is the identity component of G. It remains to show that $\Delta \subset \partial E$.

For the sake of clarity we shall now distinguish two cases.

Case 1. $G = O_n(\mathbb{C})$. Let $\Gamma \subset \Delta$ be a conjugacy class of G. In order to show that $\Gamma \subset \partial E$ it suffices to do that only in the case when $\operatorname{eig}(\Gamma) = \{-1\}$. This implies that n is even, say $n = 2m$. From [5] we infer that

$$\Gamma_{2m-2}(-1) + \Gamma_0(-1) \geqslant \Gamma,$$

and consequently we may assume that $\Gamma = \Gamma_{2m-2}(-1) + \Gamma_0(-1)$.

From Lemmas (4.1) and (4.2) we deduce that Γ lies in the closure of the union of the conjugacy classes

$$\Gamma_{m-2}(-e^{i\theta}, -e^{-i\theta}) + 2\Gamma_0(-1), \qquad 0 < \theta < \pi.$$

Since this union is contained in E, we conclude that $\Gamma \subset \bar{E}$. Since Γ is not an exponential, we have $\Gamma \subset \partial E$. Thus $\Delta \subset \partial E$.

Case 2. $G = Sp_{2n}(\mathbb{C})$. Let $\Gamma \subset \Delta$ be a conjugacy class of G. In order to show that $\Gamma \subset \partial E$, it suffices to do this only in the case when $\operatorname{eig}(\Gamma) = \{-1\}$. From [5] we infer that $\Gamma_{2n-1}(-1) \geqslant \Gamma$, and so we may assume that $\Gamma = \Gamma_{2n-1}(-1)$. From Lemmas (3.3) and (3.4) we conclude that $\Gamma \subset \bar{E}$. Since Γ is not an exponential, we have $\Gamma \subset \partial E$ and so $\Delta \subset \partial E$.

6. Proof of Theorem 4

The first assertion follows from the theorem of Borel quoted earlier.

Let $\Omega = \{a \in G \colon \tilde{q}(a) = 0\}$. We claim that Ω is open in G.

Thus let $a \in \Omega$ and let $\lambda_1, ..., \lambda_k$ be the distinct eigenvalues of a and $m_1, ..., m_k$ their respective multiplicities.

For each i, $1 \leqslant i \leqslant k$, fix a value of the argument of λ_i, say $\arg \lambda_i$. Let $\delta > 0$ be smaller than each $|\lambda_i|$. Denote by D_i the open disk in $\mathbb{C}$ centered at λ_i of radius δ. Assume that δ is small enough so that the D_i's are disjoint.

For each $\mu_i \in D_i$, $1 \leqslant i \leqslant k$, there exists a unique value $\arg \mu_i$ of its argument such that $|\arg \mu_i - \arg \lambda_i| < \pi/2$. Clearly if δ is small enough then we have

$$\sum_{i=1}^{k} m_i \, |\arg \mu_i - \arg \lambda_i| < 2\pi, \tag{6.1}$$

for arbitrary $\mu_i \in D_i$, $1 \leqslant i \leqslant k$.

Let U be a connected neighbourhood of a in G such that for each $b \in U$ the eigenvalues of b lie in the union of the disks D_i, $1 \leqslant i \leqslant k$. Since U is connected, it follows that each $b \in U$ has exactly m_i eigenvalues (counting multiplicities) in D_i. This clearly implies that $\tilde{p}(b)$ divides $\tilde{p}(a)$ for $b \in U$.

Since $\det(a) = 1$, we have

$$\sum_{i=1}^{k} m_i \arg \lambda_i = 2\pi s \tag{6.2}$$

for some integer s.

If $b \in U$ then the analog of (6.2) is valid for b, and (6.1) implies that the integer s in (6.2) will be the same for b as for a. Consequently we have $\tilde{q}(b) \equiv s \bmod \tilde{p}(b)$. Since $\tilde{q}(a) = 0$, we have $s \equiv 0 \bmod \tilde{p}(a)$ and since $\tilde{p}(b)$ divides $\tilde{p}(a)$, we conclude that $\tilde{q}(b) = 0$, i.e., $U \subset \Omega$. This proves our claim.

Since $\tilde{q}(a) = 0$ implies $q(a) = 0$, we have $\Omega \subset E$. It follows that $\Omega \subset \operatorname{Int} E$.

Let $\Delta = G \backslash \Omega$ and $a \in \Delta$. If U is any neighbourhood of a in G then U contains a non-derogatory matrix $\tilde{a}$ which has the same characteristic polynomial as a. Since $q(\tilde{a}) = \tilde{q}(a) \neq 0$, we conclude that $\tilde{a} \notin E$. Consequently $a \notin \operatorname{Int} E$ and the proof is completed.

7. Proof of Theorem 5

We prove first two lemmas.

Lemma(7.1). *Let $a \in G = U(k, k+1)$ have only one elementary divisor say $(X - \lambda)^{2k+1}$, where necessarily $|\lambda| = 1$. Then there exists a continuous function $\varphi \colon \mathbb{R} \to G$ such that $\varphi(0) = a$, $\det \varphi(t) = \lambda^{2k+1}$ for all t, and for small $t < 0$, $\varphi(t)$ has a simple eigenvalue on the unit circle.*

Proof. We may assume that G is the group of all complex matrices A of size $2k+1$ which satisfy $ASA^* = S$, where $S = S_{2k+1}$. If $\lambda = \exp(i\alpha)$ then $J_{2k+1}(i\alpha)$ belongs to the Lie algebra of G and consequently the matrix $A := \exp J_{2k+1}(i\alpha)$ belongs to G. In fact the matrices a and A are conjugate in G, and so we may assume that $a = A$.

For $t \in \mathbb{R}$ let

$$B_t = \begin{pmatrix} B_1 & v' & 0 \\ 0 & \beta & w \\ 0 & 0 & B_1 \end{pmatrix},$$

where $B_1 = J_k(i(\alpha + t))$, $v = (0, \ldots, 0, 1)$, $w = (1, 0, \ldots, 0)$, and $\beta = i(\alpha - 2kt)$. It is easy to verify that $B_t S + S B_t^* = 0$.

We now define $\varphi: \mathbb{R} \to G$ by $\varphi(t) = \exp B_t$. Clearly φ is continuous and $\varphi(0) = A = a$. The eigenvalue $\exp(i(\alpha - 2t))$ of $\varphi(t)$ is simple for small $t > 0$. It is evident that $\det \varphi(t) = \lambda^{2k+1}$ for all t.

LEMMA (7.2). *Let $a \in G = U(k, k)$ have only one elementary divisor, say $(X - \lambda)^{2k}$, where necessarily $|\lambda| = 1$. Then there exists a continuous map $\varphi: \mathbb{R} \to G$ such that $\varphi(0) = a$,* $\det$ *$\varphi(t) = \lambda^{2k}$ for all t, and for small $t > 0$, $\varphi(t)$ has a simple eigenvalue on the unit circle.*

Proof. We shall view G as a triple (G, V, f). For $\varepsilon = \pm$ we can choose a basis of V such that the matrix of f with respect to this basis is $S = \varepsilon S_{2k}$.

The Jordan block $J_{2k}(i\alpha)$, where $\lambda = \exp(i\alpha)$, satisfies

$$J_{2k}(i\alpha)\, S + S J_{2k}(i\alpha)^* = 0$$

and so $A := \exp(J_{2k}(i\alpha))$ is in G. By choosing ε appropriately, the given matrix a will be conjugate to A. Hence we may assume that $a = A$.

For $t \in \mathbb{R}$ let B_t be the matrix obtained from $J_{2k}(i\alpha)$ by replacing the zero entry in position $(k+1, k)$ with $-t^2$. The eigenvalues of B_t, $t \neq 0$, are $i\alpha$, $i(\alpha + t)$, and $i(\alpha - t)$ with respective multiplicities $2k - 2$, 1, and 1. It is easy to check that $B_t S + S B_t^* = 0$.

We now define $\varphi: \mathbb{R} \to G$ by $\varphi(t) = \exp B_t$. Clearly φ is continuous and $\varphi(0) = A = a$. All other properties required of φ also hold.

We now begin the proof of Theorem 5.

Our group $G = SU(k, n-k)$ is a subgroup of a classical group $(U(k, n-k), V, f)$, where V is an n-dimensional complex vector space and f is a non-degenerate hermitian form on V of signature $(k, n-k)$. By hypothesis, we have $2k \leqslant n$.

Condition (a) *is necessary.* Assume that $a \in G$ has an eigenvalue, say λ, on the unit circle, i.e., $|\lambda| = 1$. Then there is an orthogonal direct decomposition $V = V_1 \oplus V_2$ into a-invariant subspaces such that if $a = a_1 \oplus a_2$ is the corresponding decomposition of a then a_1 has only one elementary divisor, namely $(X - \lambda)^m$, where $m = \dim V_1$.

It follows from Lemmas (7.1) and (7.2) that there is a continuous map $\varphi: \mathbb{R} \to G$ such that $\varphi(0) = 0$ and, for $t > 0$ small, $\varphi(t)$ has a simple eigenvalue on the unit circle. It follows from [2, Theorem (3.7)] that, for $t > 0$ small, $\varphi(t)$ is an exponential in G. Hence $a \in \bar{E}$, which proves the necessity of (a).

Condition (b) *is necessary.* Let $a \in \operatorname{Ext} E$. We know that a has no eigenvalues on the unit circle. Since the eigenvalues of a are

$$\lambda_1, \bar{\lambda}_1^{-1}, \ldots, \lambda_k, \bar{\lambda}_k^{-1},$$

with corresponding multiplicities

$$m_1, m_1, ..., m_k, m_k,$$

and since $\det(a) = 1$, it follows that the number s defined in condition (b) is indeed an integer.

Choose $\delta > 0$ small so that all open disks

$$D_l := \{z \in \mathbb{C}: |z - \lambda_l| < \delta\}$$

lie inside the unit circle, do not contain the origin, and are pairwise disjoint. If $\mu_l \in D_l$, $1 \leqslant l \leqslant k$, we denote by $\arg \mu_l$ the unique value of the argument of μ_l which satisfies the inequality

$$|\arg \mu_l - \arg \lambda_l| < \pi/2.$$

We also assume that δ has been chosen small enough so that $\mu_l \in D_l$, $1 \leqslant l \leqslant k$, imply that

$$\sum_{l=1}^{k} m_l \, |\arg \mu_l - \arg \lambda_l| < \pi. \tag{7.3}$$

Now let $U \subset \operatorname{Ext} E$ be a neighbourhood of a in G such that each $b \in U$ has precisely m_l eigenvalues (counting multiplicities) in D_l, $1 \leqslant l \leqslant k$. It follows from (7.3) that the integer s defined in condition (b) for the element a remains the same when a is replaced with any $b \in U$. We can choose $b \in U$ so that all eigenvalues of b are simple. Since $b \notin E$, Theorem (3.7) of [2] implies that s is odd.

Sufficiency. Let Ω be the set of all matrices in G which satisfy the conditions (a) and (b). By [2, Theorem (3.7)] we know that $\Omega \cap E = \varnothing$. From the proof of the necessity of (b) it is clear that Ω is open in G. Consequently we have $\Omega \subset \operatorname{Ext} E$.

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